Math 1552

Sections 10.3, 10.4 and 10.5 : Convergence Tests for Infinite Series

Math 1552 lecture slides adapted from the course materials
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Learning Goals

"tests"

- Learn how to apply the integral, comparison, limit comparison, ratio and root series to determine whether an infinite series converges or diverges
- Learn when to apply which test
- Summarize the results into a formal mathematical justification

Quick review…

- The harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ DIVERGES. (important to remember)

even though $\lim_{k \to \infty} \frac{1}{k} = 0$, the series still diverges

Quick review…

- The harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ DIVERGES.
- Telescoping series CONVERGE. Find the sum using partial fraction decompositions.

$$\sum_{N=0}^{\infty} \frac{1}{(N+a)(N+b)}$$

Quick review…

- The harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ DIVERGES.
- Telescoping series CONVERGE. Find the sum using partial fraction decompositions.
- A geometric series

$$\sum_{k=0}^{\infty} r^k \qquad \begin{array}{l} \textit{converges} \text{ to } \dfrac{1}{1-r} \text{ when } |r|<1 \\ \textit{diverges} \text{ when } |\mathrm{r}| \geq 1 \end{array}$$

$-1 < r < 1$

$\longrightarrow$ when $r \leq -1$ or $r \geq 1$

Divergence (n^{th} term) Test

Sequence – a_n

Series – $\sum_{n=0}^{\infty} a_n$

Given $\sum_{k=0}^{\infty} a_k$, first find $\lim_{n\to\infty} a_n$.

If $\lim_{n\to\infty} a_n \neq 0$, then the series **DIVERGES!**

Otherwise, the test is *INCONCLUSIVE*

and you must try another test.

↳ we will have many tests to choose from

Integral Test

Let f be a continuous, positive, and decreasing function. Then:

$$\sum_{k=1}^{\infty} f(k)\ \textit{converges}\ \text{if and only if} \int_1^{\infty} f(x)dx\ \text{converges,}$$

$$\text{and } \textit{diverges} \text{ if and only if } \underbrace{\int_1^{N} f(x)dx \to \infty \text{ as } N \to \infty.}$$

$\int_1^{\infty} f(x)dx$ diverges

OR $\lim_{N\to\infty} \int_1^{N} f(x)dx = +\infty$ (diverges)

Example I:

Use the integral test to determine whether the series converges: $\sum_{k=2}^{\infty} \frac{1}{k \ln k} = S$

$\rightarrow$ S converges if $I = \int_2^\infty \frac{dx}{x \ln(x)}$ converges, and diverges if I diverges.

(apply the integral test to $f(k) = \frac{1}{k \ln(k)}$:

- positive for $k \geq 2$
- decreasing
- continuous)

→ so we need to evaluate I

$$I = \lim_{b \to \infty} \int_2^b \frac{dx}{x \ln(x)}$$

→ evaluate the indefinite integral:

u-sub: $u = \ln(x)$

$$du = \frac{dx}{x}$$

$$\int \frac{dx}{x\ln(x)} = \int \frac{du}{u} = \ln|u| + C$$

$$= \ln|\ln x| + C$$

$$I = \lim_{b\to\infty} \left(\ln|\ln x|\right)\Big|_2^b$$

$$= \underbrace{\lim_{b\to\infty} \ln|\ln b|}_{} - \ln|\ln(2)|$$

as $b \to \infty$ $|\ln b| \to \infty$

so the limit is the same as

$$\lim_{x\to\infty} \ln(x) = +\infty$$

So $I = +\infty$, and the series <u>diverges</u> by the integral test.

Example II:

When does a p-series converge? $S_p = \sum_{k=1}^{\infty} \frac{1}{k^p}$ (p - series)

harmonic series
is a p-series
with $p=1$
(recall)

$\rightarrow f(k) = \frac{1}{k^p}$: • cts
• decreasing
• positive

$\rightarrow$ then S_p will converge when
$I_p = \int_1^{\infty} \frac{dx}{x^p}$ converges, and diverge
when $I_p = +\infty$ (diverges)

$\rightarrow I_p = \lim_{b \to \infty} \int_1^b \frac{dx}{x^p}$

→ let's look at three cases:

(1) $p=1$

(2) $p<1$

(3) $p>1$

Case (1): $I_1 = \lim_{b\to\infty} \int_1^b \frac{dx}{x} = \lim_{b\to\infty} \ln|b| - \cancel{\ln|1|}^{0}$

$= +\infty$

So by the integral test, S_1 diverges

(S_1 is the same as the harmonic series ✓)

Case (2): $(p<1)$

$$I_p = \lim_{b\to\infty} \overbrace{\int_1^b \frac{dx}{x^p}}^{=\left.\frac{x^{1-p}}{1-p}\right|_1^b} = \lim_{b\to\infty} \frac{1}{1-p}\cdot \underbrace{\frac{1}{b^{p-1}}}_{\to\, b^{-(p-1)}} \underbrace{- \frac{1}{1-p}}_{\text{some constant}}$$

when $p<1$, $p-1<0 \iff -(p-1)>0$

so the limit is: $\dfrac{1}{1-p} \lim\limits_{x\to\infty} x^{\text{something positive}}$

$= +\infty$

So in this case, if $p<1$, S_p diverges by the integral test.

<u>Case (3):</u> $(p>1)$

$$I_p = \lim_{b\to\infty}\int_1^b \frac{dx}{x^p} = \lim_{b\to\infty} \frac{1}{(1-p)x^{p-1}}\Bigg|_1^b$$

$$= \frac{1}{1-p}\left[\underbrace{\lim_{b\to\infty}\frac{1}{b^{p-1}}}_{=0} - 1\right]$$

So I_p converges when $p>1$, and so by the integral test, S_p converges when $p>1$

Summary:

→ a p-series is an infinite series of the form $\sum_{k=1}^{\infty} \frac{1}{k^p}$

→ a p-series converges when $p > 1$, and diverges for any $p \leq 1$.

(use this as a fact in combination with other tests for converge)

Series we know:

- The harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ DIVERGES. (a p-series for p=1)
- A geometric series

$$\sum_{k=0}^{\infty} r^k \qquad \begin{array}{l} \textit{converges} \text{ to } \dfrac{1}{1-r} \text{ when } |r|<1 \\ \textit{diverges} \text{ when } |\mathrm{r}| \geq 1 \end{array}$$

- A p-series

$$\sum_{k=1}^{\infty} \frac{1}{k^p} \qquad \begin{array}{l} \textit{converges when } p>1 \\ \textit{diverges} \text{ when } p \leq 1 \end{array}$$

(Put this on your short review sheet of topics we have seen so far for quiz 3)

Some Convergence Theorems

(1) If $\sum \overset{A}{a_k}$ and $\sum \overset{B}{b_k}$ both converge, then $\sum (a_k \pm b_k)$ also converges.

if A, B converge, then $\sum_k (a_k \pm b_k)$ converges $(A \pm B)$

(2) If $\sum \overset{A}{a_k}$ converges, then $\sum c a_k$ also converges for any $c \in \Re$.

(to $c \cdot A$)

some constant

(3) If $\sum_{k=j}^{\infty} a_k$ converges, so does $\sum_{k=0}^{\infty} a_k$.

A_J

$$\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{J-1} a_k + A_J = \text{finite sum} + A_J$$

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Section 10.4: Comparison Tests for Infinite Series

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Recap of last class:

e.g., $\lim_{k\to\infty} a_k \neq 0$

- *Divergence test*: if the limit is not 0, the series diverges (nth term test)
- *Integral test*: use with a function that has an "easy" antiderivative

$$\sum_{k=2}^{\infty} \frac{1}{k \ln k}$$

f

conditions:
f is positive, cts, and
decreasing

Basic Comparison Test: $\boxed{\text{Part (a)}}$

Let $\sum\limits_k \overset{A}{a_k}$ be a series with $a_k \geq 0$ for all k.

If we can find a series $\overset{C}{\sum\limits_k} c_k$ such that

$\sum\limits_k c_k$ converges and $0 \leq a_k \leq c_k$ for all but

finitely many terms, then $\sum\limits_k a_k$ must also

converge.

Means: if for all large k we have $0 \leq a_k \leq c_k$, and $C < \infty$, then $0 \leq A \leq C < \infty$, e.g., $A < \infty$ (converges).

Basic Comparison Test: $\boxed{\textit{Part (b)}}$

Let $\overset{A}{\sum_{k}} a_k$ be a series with $a_k \geq 0$ for all k.

If we can find a series $\overset{D}{\sum_{k}} d_k$ such that

$\sum_{k} d_k$ diverges and $a_k \geq d_k \geq 0$ for all but

finitely many terms, then $\sum_{k} a_k$ must also

diverge.

<u>Means</u>: if for all large k we have that $a_k \geq d_k \geq 0$
and $D = +\infty$ (diverges), then since $A \geq D = +\infty$,
$A = +\infty$ (diverges)

Example: Does this series converge?

$$\text{(A)} \sum_{k=1}^{\infty} \frac{1}{1+2^k} = S$$

$\rightarrow$ try a basic comparison test

$\rightarrow$ $\displaystyle\sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{1}{2} \cdot \frac{1}{1-1/2} = 1$ (converges as a geometric series with $r = 1/2$)

$\rightarrow$ define: $a_k = \dfrac{1}{1+2^k}$, $b_k = \dfrac{1}{2^k}$

$$1+2^k \geq 2^k \quad \text{for } k \geq 1$$

$$\frac{1}{2^k} \geq \frac{1}{1+2^k} \quad \text{for } k \geq 1$$

→ so basic comparison test part (a) tells us that S converges if

$$\sum_{k=1}^{\infty} b_k \text{ converges } \checkmark$$

→ so the series S converges.

Example: Does this series converge?

(B) $\sum_{k=2}^{\infty} \frac{1}{\sqrt{k}-1} = S$ (looks almost like a p-series with $p = 1/2$, so expect to diverge?)

→ let's try to apply a basic comparison test

→ Note that $\sum_{k=2}^{\infty} \frac{1}{\sqrt{k}}$ diverges as a p-series with $p = 1/2$

→ <u>define</u>: $a_k = \frac{1}{\sqrt{k}-1}$, $d_k = \frac{1}{\sqrt{k}}$

→ Notice that $\sqrt{k} \geq \sqrt{k} - 1$ for $k \geq 2$

this means that $\dfrac{1}{\sqrt{k}-1} \geq \dfrac{1}{\sqrt{k}}$ for $k \geq 2$

or: $a_k \geq d_k$ for $k \geq 2$

→ we have that $\sum_{k=2}^{\infty} d_k = +\infty$ (diverges),

by the basic comparison test part (b),

the series $S = \sum_{k=2}^{\infty} a_k$ diverges.